Exam for "Symmetries and quantum field theory"
Master 2 ICFP quantum physics
Teachers : J.N. Fuchs and G. Orso
Januray 5^{th} 2017 from 14 : 00 till 17 : 00

Duration : 3h. Handwritten notes, typed lecture notes and typed sheets given during the lectures/exercise sessions are the only documents allowed (no books, computers, phones, etc.).

The two exercises are independent.

Exercise A : Charge conjugation and Majorana bi-spinor

In the following it will be important to distinguish a spinor (with 2 components) and a bi-spinor (with 4 components).

1. Consider a Dirac bi-spinor ψ in 3+1 spacetime. We define charge conjugation as the following transformation

$$\psi \to \psi^c = -i\gamma^2\psi^* \tag{1}$$

in either the chiral or the standard representation of the gamma matrices (this definition may not be valid in another representation). What is $(\psi^c)^c$?

2. If the Dirac field $\psi(x)$ satisfies the following equation

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu(x) - m)\psi(x) = 0 \tag{2}$$

in the presence of a gauge field A^μ, what is the equation satisfied by $\psi^c(x)$?

3. Let χ_L be a left Weyl spinor. We define the corresponding left Weyl bi-spinor in the chiral representation as :

$$\chi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}. \tag{3}$$

Show that χ is an eigenvector of the chirality operator γ^5. Show that

$$\chi^c = \begin{pmatrix} 0 \\ i\sigma_y\chi_L^* \end{pmatrix}. \tag{4}$$

Is it also an eigenvector of γ^5 ? A Weyl bi-spinor can be defined as a bi-spinor χ satisfying a "chirality condition" $\gamma^5\chi = \pm\chi$.

Knowing that χ_L transforms with the matrix Λ_L under a Lorentz transformation, show that $i\sigma_y\chi_L^*$ indeed transforms as a right Weyl spinor.

4. Given a left Weyl spinor ψ_L, we define a Majorana bi-spinor ψ_M in the chiral representation as :

$$\psi_M = \begin{pmatrix} \psi_L \\ i\sigma_y\psi_L^* \end{pmatrix}. \tag{5}$$

Compute ψ_M^c as a function of ψ_M. This equation is called a "reality condition". Show that it is Lorentz invariant. Starting from the following left Weyl spinor

$$\psi_L = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{6}$$

with α and β two complex numbers, give the general expression of a Majorana bi-spinor in the chiral representation.

From now on, we take as a definition of a Majorana bi-spinor that it is a bi-spinor that satisfies the reality condition derived below equation (5).

5. A Dirac bi-spinor has 4 independent complex components. A Weyl bi-spinor has 2 independent complex components. How many independent complex components does a Majorana bi-spinor have ?

6. Consider now a Majorana field $\psi_M(x)$ satisfying the following Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi_M(x) = 0. \tag{7}$$

Using equation (5), write the two corresponding equations on $\psi_L(x)$. Are these equations independent ?

7. Show that the image $\psi'_M = e^{i\theta}\psi_M$ of a Majorana bi-spinor ψ_M by a $U(1)$ transformation is not a Majorana bi-spinor. In the chiral representation, the Lagrangian describing a Majorana field (5) can be taken as :

$$\mathcal{L} = \psi_L^\dagger i\bar{\sigma}^\mu \partial_\mu \psi_L - \frac{m}{2}[\psi_L^\dagger (i\sigma_y \psi_L^*) + (i\sigma_y \psi_L^*)^\dagger \psi_L]. \tag{8}$$

Check explicitly that the corresponding action is not $U(1)$ invariant. Show that the Majorana mass term $\psi_L^\dagger (i\sigma_y \psi_L^*) + (i\sigma_y \psi_L^*)^\dagger \psi_L$ vanishes unless the Weyl spinor ψ_L is made of anti-commuting numbers (known as Grassman variables).

8. Can a Majorana field couple to the electromagnetic gauge field ? What is its Dirac equation in the presence of a gauge field ?

9. We now discuss another representation of gamma matrices (beyond the chiral and standard ones) called the Majorana representation (1937). It reads

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \; \tilde{\gamma}^1 = \begin{pmatrix} i\sigma_x & 0 \\ 0 & i\sigma_x \end{pmatrix}, \; \tilde{\gamma}^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \; \tilde{\gamma}^3 = \begin{pmatrix} i\sigma_z & 0 \\ 0 & i\sigma_z \end{pmatrix} \tag{9}$$

Show that, in this representation, the Dirac equation $(i\tilde{\gamma}^\mu \partial_\mu - m)\tilde{\psi}(x) = 0$ can have solutions which are real $\tilde{\psi}(x)^* = \tilde{\psi}(x)$. This is the reality condition in the Majorana representation. What could be the charge conjugation in the Majorana representation ? Compute $\tilde{\gamma}^5$ and show that a Majorana field can not be a chirality eigenvector.

10. Neutrinos are spin $1/2$ particles that are charge neutral and have a small but finite mass. What type of fermonic field (Dirac, Weyl or Majorana) could be used to describe neutrinos ?

11. Show that a Dirac bi-spinor can be written as the sum of a left Weyl bi-spinor and an independent right Weyl bi-spinor. Show that it can also be written as a Majorana bi-spinor plus $i = \sqrt{-1}$ times another Majorana bi-spinor.

References :

P.B. Pal, *Dirac, Majorana, and Weyl fermions*, Am. J. Phys. **79**, 485 (2011). https://arxiv.org/abs/1006.1718

Exercise B : Massless Dirac equation in 1+1 spacetime

Relativistic physics : chiral anomaly in 1+1 dimension

1. Consider a Dirac spinor ψ in 1+1 spacetime (we take $\hbar = 1$ and $c = 1$ for the first nine questions). Show that the Clifford algebra can be satisfied with 2×2 matrices. In the following we choose $\gamma^0 = \sigma_x$ and $\gamma^1 = i\sigma_y$ as the chiral representation. Justify this choice.

2. Find the chirality matrix γ^5 such that it anticommutes with all other gamma matrices, squares to one and is hermitian. Check that a left and a right Weyl spinor have the correct chirality eigenvalues. In the following, we will call them left and right movers.

3. We first consider a massless Dirac equation. Write it in the chiral basis such that

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{10}$$

What plays the role of $\bar{\sigma}^\mu$ and σ^μ in 1+1 spacetime ?

4. Write the 1+1 Weyl equation for a right field. What is the corresponding dispersion relation ? And for a left field ?

5. Write the Weyl equation for a right field in the presence of a gauge field A^μ with the following gauge choice :

$$A^\mu = (0, -E_x t) \tag{11}$$

corresponding to the presence of an external electric field E_x.

6. Solve this equation with the following ansatz

$$\psi_R(t, x) = \psi_0 e^{ikx} e^{-i\int^t dt' \omega(t')} \tag{12}$$

where ψ_0 is a constant. Find $\omega(t)$. What is the equation satisfied by $\dot{\omega}(t)$?

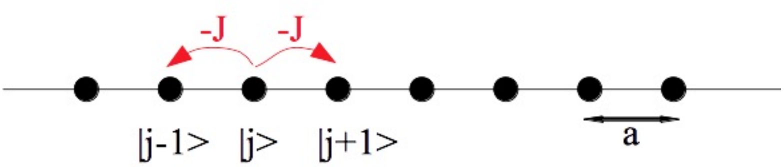
7. Now, consider a right Weyl field and a situation where at $t = 0$ all negative energy modes are filled. What is the effect of the electric field on this sea of Weyl particles ? Assume that the system is on a ring of length L. What is the rate of change of the density $n_R = N_R/L$ of right movers ?

8. Similarly get the rate of change $\dot{n}_L$ of left movers. Give the rate of change of $n_R + n_L$ and $n_R - n_L$.

9. Starting again from the massless Dirac equation in 1+1 dimension, show that there are two conserved currents related to internal symmetries. What are the corresponding conserved charges ? Comment on the fact that in question 8., you found that $n_R + n_L$ is conserved but not $n_R - n_L$. This non-conservation of right and left movers is called a chiral anomaly.

Solid state physics : tight-binding model on an atomic chain

In this section, we consider a condensed matter system, whose low energy behavior mimicks the above relativistic physics.



10. Consider a tight-binding model for the motion of an electron in an atomic chain (a one-dimensional crystal). Here we do NOT take $\hbar = 1$ and $c = 1$ as we consider non-relativistic physics. The atoms are

equally separated by a distance a and each carries a single atomic orbital $|j\rangle$ with $j \in \mathbb{Z}$. We assume that the single-particle Hilbert space (we are now doing quantum mechanics) is spanned by the following orthonormal basis $\{|j\rangle, j \in \mathbb{Z}\}$. The single-particle Hamiltonian is

$$H = -J \sum_j (|j+1\rangle\langle j| + |j\rangle\langle j+1|) \tag{13}$$

where $J > 0$ is the tunneling amplitude between neighboring atomic orbitals. Write the stationary Schrödinger equation and find the corresponding eigenvalues $\varepsilon(q)$ and eigenvectors $|q\rangle$. Draw the dispersion relation $\varepsilon(q)$.

11. The energy band is half-filled, i.e. each atom contributes half an electron to the chain (we do not consider spin degeneracy and consider electrons as if they were spinless). On the dispersion relation $\varepsilon(q)$ drawn before, indicate the Fermi energy ε_F. What is the Fermi momentum q_F ? Linearize the dispersion relation in the vicinity of the two Fermi points $q = \pm q_F + k$ and give the corresponding velocity v.

Hint : if needed, consider a finite-size system with periodic boundary conditions : N atoms are on a ring of length L.

12. Write the corresponding Weyl equations near q_F and $-q_F$. What is the validity of this effective description ?

13. Going back to the full tight-binding model (not linearized), discuss the effect of an applied electric field E_x on the Fermi sea (the groundstate). Comment on the chiral anomaly in this case (i.e. the non-conservation of left and right movers). What is the name of this phenomenon in solid-state physics ?

References :

A. Zee, "Quantum field theory in a nutshell" (Princeton university press, 2003, first edition), chapter V.5, pages 273-274.

H.B. Nielsen and M. Ninomiya, Phys. Lett. **130B**, 389 (1983).