Exam for "Symmetries and quantum field theory"
Master 2 ICFP quantum physics
Teachers : J.N. Fuchs and G. Orso
December 21$^{\text{th}}$ 2017 from 14 : 00 till 17 : 00

Duration : 3h. Handwritten notes, typed lecture notes and typed sheets given during the lectures/exercise sessions are the only documents allowed (no books, computers, phones, etc.).

Problem : Dirac equation and charge fractionalisation

I. 1D chain

Consider a tight-binding model for the motion of an electron in an atomic chain (a one-dimensional crystal), see Figure 1. The atoms are equally spaced by a distance a and each carries a single atomic orbital $|j\rangle$ with $j \in \mathbb{Z}$. The position of atoms is $x = ja$. We assume that the single-particle Hilbert space (we are doing quantum mechanics at the moment) is spanned by the following orthonormal basis $\{|j\rangle,\, j \in \mathbb{Z}\}$. The single-particle Hamiltonian is

$$H = -J \sum_j \left(|j+1\rangle\langle j| + |j\rangle\langle j+1|\right) \tag{1}$$

where $J > 0$ is the hopping amplitude between neighboring atomic orbitals.

We take units such that $a = 1$, $J = 1$ and $\hbar = 1$.

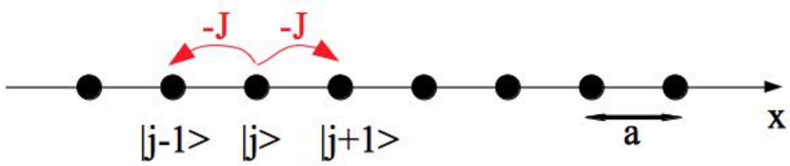

FIGURE 1 – One-dimensional chain.

1. Write the stationary Schrödinger equation and find the corresponding eigenvalues $\varepsilon(q)$ and eigenfunctions $\langle j|q\rangle$. Draw the dispersion relation $\varepsilon(q)$. In which interval is the wavevector q ?

2. The energy band is half-filled, i.e. each atom contributes half an electron to the chain (we do not consider spin degeneracy and treat electrons as if they were spinless). Give the Fermi energy ε_F and momentum q_F and indicate them on the dispersion relation $\varepsilon(q)$ drawn before. Linearize the dispersion relation in the vicinity of the two Fermi points $q = \pm q_F + k$ and give the corresponding velocity v [restoring the units of a, J and $\hbar$]. Draw the linearized dispersion relation on the same plot as $\varepsilon(q)$.
Hint : if needed, consider a finite-size system with periodic boundary conditions : N atoms, with N even, are on a ring of length L containing $N/2$ electrons.

3. Linearize the time-dependent Schrödinger equation near q_F and $-q_F$ to obtain two Weyl equations. Call ψ_L and ψ_R the corresponding wavefunctions. What is the range of validity of this effective description ?

4. Write the two Weyl equations in a single Dirac equation in the chiral representation where $\psi = (\psi_L, \psi_R)^T$ is a doublet of complex numbers. Give both the γ^μ matrices and the α_x and β matrices in terms of Pauli matrices. Show that chirality is conserved (i.e. that γ_5 commutes with the Dirac Hamiltonian). What is the intuitive meaning of chirality here ?

II. 1D chain with staggered potential

5. We now consider a 1D chain with a staggered on-site potential $\pm\Delta$. To the tight-binding Hamiltonian (1), we add the term

$$H_{\text{staggered}} = \sum_j \varepsilon_j |j\rangle\langle j| \,, \tag{2}$$

where $\varepsilon_j = \Delta$ if j is odd (A site) and $\varepsilon_j = -\Delta$ if j is even (B site), with $\Delta \in \mathbb{R}$. Draw the chain and indicate the two sublattices A and B. What is the periodicity of this crystal ? What is the first Brillouin zone ? Find the dispersion relation and plot it. Each atom still contributes half an electron : is this crystal a conductor or an insulator at zero temperature ?

6. Write the low-energy Dirac Hamiltonian in the vicinity of $q = \pi/2$ (why did we not choose $q = -\pi/2$?) and give its mass [restoring the units of J, a and $\hbar$]. To do so, it is more convenient to work in the sublattice representation $\psi = (\psi_A, \psi_B)^T$ [in the following, we will stick to this sublattice representation]. The corresponding Dirac matrices will now be indicated with a tilde $\tilde{\gamma}^\mu$ in order to distinguish them from those γ^μ in the chiral representation. Give also the $\tilde{\alpha}_x$ and $\tilde{\beta}$ matrices. Is chirality conserved ?

III. Domain wall and zero-energy mode

The staggered on-site potential [$\varepsilon_{j\text{ odd}} = \Delta$ and $\varepsilon_{j\text{ even}} = -\Delta$] actually results from spontaneous symmetry breaking of some other degrees of freedom that we do not specify here (these could be lattice vibrations for example). There is a high-temperature symmetric phase, with $\mathbb{Z}_2$ symmetry and $\Delta = 0$. And a low-temperature ordered phase, with spontaneously broken $\mathbb{Z}_2$ symmetry and $\Delta \neq 0$. In the ordered phase, there are two equally likely possibilities : either $\Delta > 0$ or $\Delta < 0$. Up to this point, we have considered the case that either $\Delta > 0$ or $\Delta < 0$ everywhere in space, i.e. a homogeneous groundstate. We now consider an inhomogeneous groundstate such that there is a domain wall at $x \sim 0$ between a region at negative x with $\Delta < 0$ and a region at positive x with $\Delta > 0$, see Figure 2. For concreteness, you may think that $\Delta(x) = |\Delta|\text{sign}(x)$, which is an abrupt domain wall, or that $\Delta(x) = |\Delta|\tanh(x/l)$, which is a smooth domain wall with characteristic size $l \sim 10a$, for example. Such a domain wall is also known as a kink.

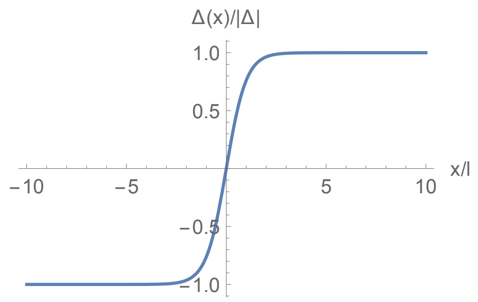

FIGURE 2 – Domain wall (kink) at $x = 0$.

In the following, we solve the stationary Dirac equation in the presence of the kink $\Delta(x) = |\Delta|\text{sign}(x)$.

7. Show that the Dirac Hamiltonian anticommutes with σ_y so that the energy spectrum necessarily has the $\varepsilon \to -\varepsilon$ symmetry.

8. Write the stationary Dirac equation in the presence of the kink. There are delocalized solutions both at positive and at negative energy. We respectively call them $\psi_{k+}(x,t) = \psi_{k+}(x)e^{-i\varepsilon_k t}$ and $\psi_{k-}(x,t) = \psi_{k-}(x)e^{i\varepsilon_k t}$. What is the corresponding dispersion relation $\pm\varepsilon_k$?
Hint : As in a scattering problem with a piecewise constant potential, write the solution $\psi(x)$ separately in each domain $x > 0$ and $x < 0$ and impose continuity of ψ at $x = 0$.

9. In addition to the above continuum energy spectrum with delocalized eigenfunctions, there is also a

discrete localized solution. Consider the following ansatz :

$$\psi_0(x) = f(x) \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{3}$$

Check that it is an eigenvector of σ_y. Assuming that the ansatz is an eigenstate of the Hamiltonian, show that the corresponding energy is zero. Then calculate $f(x)$ explicitly.

10. Is the zero-energy solution square-integrable ? What is its localization length ξ [restoring the units of J, a and $\hbar$] ?

11. Show that the charge conjugation transformation is $\psi \to \psi^c = i\sigma_x\psi^*$ in the sublattice representation $\tilde{\gamma}^\mu$. What is the behavior of the zero-energy mode ψ_0 under charge conjugation ?

In the end, we assume that $\{\psi_0(x), \psi_{k+}(x), \psi_{k-}(x), \forall k\}$ form an orthonormalized basis of solutions for the Dirac equation in the presence of a kink.

IV. Quantum field theory and charge fractionalisation

In this section, we put hats on Fock space operators.

12. We now consider the quantum theory for the Dirac field in the presence of a domain wall. From the solutions of the stationary Dirac equation , the quantum field is

$$\hat{\Psi}(x,t) = \hat{a}\psi_0(x) + \sum_k \left(\hat{b}_k\psi_{k+}(x)e^{-i\varepsilon_k t} + \hat{d}_k^\dagger\psi_{-k-}(x)e^{i\varepsilon_k t}\right) \tag{4}$$

as a function of the operators $\hat{b}_k$ (positive energy modes), $\hat{d}_k$ (negative energy modes) and $\hat{a}$ (zero-energy mode). From the anticommutation relation for $\hat{\Psi}$, show that the algebra for the $\hat{a}$ operator is the usual one for a fermionic operator.

13. Write the Hamiltonian operator $\hat{H}$ using the normal order prescription as a function of $\hat{b}_k$, $\hat{d}_k$ and $\hat{a}$. Show that the groundstate of the quantum field theory is actually twofold degenerate as the zero-energy mode can be either empty or occupied (by an electron) without changing the total energy. We call the two groundstates $|\text{empty}\rangle$ and $|\text{occupied}\rangle$ and define their energy to be $E_0 = 0$. What is the action of $\hat{a}$ and $\hat{a}^\dagger$ on $|\text{empty}\rangle$ and $|\text{occupied}\rangle$?

14. Write the charge operator $\hat{Q}$ in terms of $\hat{b}_k$, $\hat{d}_k$ and $\hat{a}$ using the normal order prescription. Due to the presence of the zero-energy mode, this prescription is such that $\hat{Q}$ breaks the symmetry between particles and anti-particles. Modify the charge operator $\hat{Q}$ (i.e. choose another prescription) such that it does not favor either particles or anti-particles. Then show that $|\text{occupied}\rangle$ has eigenvalue $Q = 1/2$, whereas $|\text{empty}\rangle$ has eigenvalue $Q = -1/2$. Is it surprising that Q takes non-integer values ?

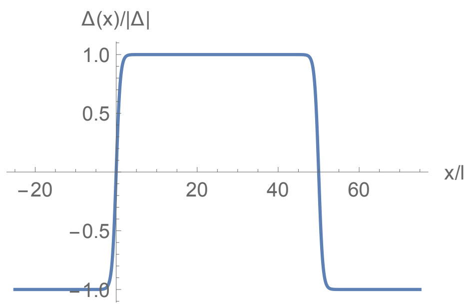

FIGURE 3 – A kink at $x = 0$ and an anti-kink at $x = 50$ for a ring of $N = 100$ sites.

15. In a system with periodic boundary conditions (i.e. a ring of length L), it is not possible to have a single domain wall. Domain walls necessarily come in pairs : a kink and an anti-kink, something like

$\Delta(x) = |\Delta|[\text{sign}(x) - \text{sign}(x - L/2) - 1]$, see Figure 3. How many zero-modes are present on such a ring [we assume that the kink and anti-kink are far apart, such that $L/2 \gg \xi$] ? Consider a ring with an even number of sites N that is half-filled $N_{\text{electron}} = N/2$. How is it possible to simultaneously have an integer number of electrons and fractional charges ? How many groundstates (for the QFT) at $E_0 = 0$ are possible ? What is their charge Q ?

V. Mass-kink in a 2+1 Dirac equation

Here we take units such that $v = 1$ and $\hbar = 1$.

16. As an extension of the previous model, we now consider a 2+1 Dirac equation with a constant mass term m. Keeping the $\tilde{\alpha}_x$ and $\tilde{\beta}$ matrices from the 1+1 case, give $\tilde{\alpha}_y$ and write the Dirac Hamiltonian. What is the dispersion relation ?

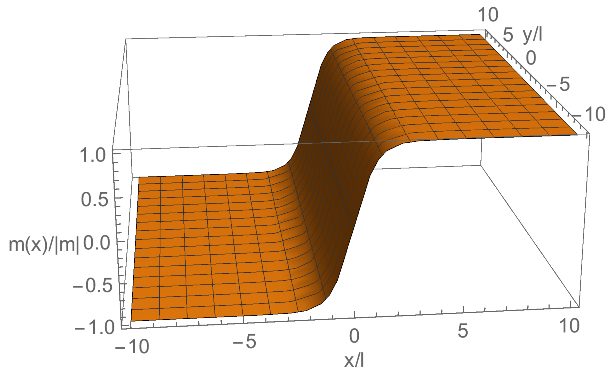

FIGURE 4 – A kink in the mass of a 2+1 Dirac equation.

17. We now consider an inhomogeneous mass term $m(x, y) = |m|\text{sign}(x)$, i.e. a mass-kink, see Figure 4. Write the 2D eigenvalue equation and use the fact that the Dirac Hamiltonian commutes with the momentum operator p_y to reduce it to an effective 1D eigenvalue equation parametrized by its eigenvalue k_y.

18. Inspired by the above zero-energy ansatz (3), find a branch of solutions in the vicinity of $\varepsilon = 0$ that is parametrized by k_y. What is the velocity of this excitation mode ? Comment its direction of motion. What happens if the kink is replaced by an anti-kink ?

References :

R. Jackiw, “Fractional and Majorana fermions : the physics of zero energy modes”, arXiv :1104.4486 .
R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).