Exercise A : Charge conjugation and Majorana bi-spinor

1) $(\psi^c)^c = -i\gamma^2(\psi^c)^* = -i\gamma^2(-i\gamma^2\psi^*)^* = -i\gamma^2 i(\gamma^2)^*\psi = \gamma^2(\gamma^2)^*\psi$

but $\gamma^2(\gamma^2)^* = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix} = \mathbb{1}_4 \Rightarrow (\psi^c)^c = \psi$

2) $(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi = 0 \xrightarrow{*} (-i(\gamma^\mu)^*\partial_\mu + e(\gamma^\mu)^* A_\mu - m)\psi^* = 0$

insert $(-i\gamma^2)^*(-i\gamma^2) = \mathbb{1}_4$ and multiply on the left by $(-i\gamma^2) \rightarrow$

$(-i\gamma^2)\left[-i(\gamma^\mu)^*\partial_\mu + e(\gamma^\mu)^* A_\mu - m\right](-i\gamma^2)^*\underbrace{(-i\gamma^2\psi^*)}_{\psi^c} = 0$

but $(-i\gamma^2)(\gamma^\mu)^*(-i\gamma^2)^* = -\gamma^2(\gamma^\mu)^*\gamma^2 = -\gamma^\mu$ and $(-i\gamma^2)(-i\gamma^2)^* = \mathbb{1}_4$

therefore $\left[i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\right]\psi^c = 0$ ie. Dirac eqt with opposite electric charge

summary: $\left[i\gamma^\mu D_\mu - m\right]\psi = 0 \rightarrow \left[i\gamma^\mu D_\mu^* - m\right]\psi^c = 0 \qquad D_\mu \equiv \partial_\mu - ieA_\mu$

3) $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ in the CR.

$\gamma^5\chi = \gamma^5\begin{pmatrix} \chi_L \\ 0 \end{pmatrix} = -\chi_L$

$\chi^c = -i\gamma^2\chi^* = -i\begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}\begin{pmatrix} \chi_L^* \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i\sigma_y\chi_L^* \end{pmatrix} \qquad \gamma^5\chi^c = (+1)\chi^c$

$\chi_L \xrightarrow{\Lambda} \chi_L' = \Lambda_L\chi_L \qquad i\sigma_y\chi_L^* \xrightarrow{\Lambda} i\sigma_y(\Lambda_L\chi_L)^* = i\sigma_y\Lambda_L^*\chi_L^*$

$= i\sigma_y\Lambda_L^*\sigma_y(\sigma_y\chi_L^*) = \underbrace{\sigma_y\Lambda_L^*\sigma_y}_{\Lambda_R}(i\sigma_y\chi_L)^*$

4) $\psi_M^c = -i\gamma^2\psi_M^* = -i\begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}\begin{pmatrix} \psi_L^* \\ -i\sigma_y^*\psi_L \end{pmatrix} = -i\begin{pmatrix} i\sigma_y^2\psi_L \\ -\sigma_y\psi_L^* \end{pmatrix} = \begin{pmatrix} \psi_L \\ i\sigma_y\psi_L^* \end{pmatrix} = \psi_M$

$\psi_M^c = \psi_M$ is a "reality condition"

If $\psi_M^c = \psi_M$ and $\psi_M \xrightarrow{\Lambda} \psi_M' = \begin{pmatrix} \Lambda_L\psi_L \\ \Lambda_R i\sigma_y\psi_L^* \end{pmatrix}$ do we have $(\psi_M')^c = \psi_M'$?

$(\psi_M')^c = -i\gamma^2\begin{pmatrix} \Lambda_L\psi_L \\ \Lambda_R i\sigma_y\psi_L^* \end{pmatrix}^* = -i\begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}\begin{pmatrix} \Lambda_L^*\psi_L^* \\ \Lambda_R^* i\sigma_y\psi_L \end{pmatrix} = -i\begin{pmatrix} \sigma_y\Lambda_R^*\sigma_y i\psi_L \\ -\sigma_y\Lambda_L^*\psi_L^* \end{pmatrix}$

but $\sigma_y\Lambda_R^*\sigma_y = \Lambda_L$ and $\sigma_y\Lambda_L^* = \Lambda_R\sigma_y$

$\rightarrow (\psi_M')^c = -i\begin{pmatrix} \Lambda_L i\psi_L \\ -\Lambda_R\sigma_y\psi_L^* \end{pmatrix} = \begin{pmatrix} \Lambda_L\psi_L \\ \Lambda_R i\sigma_y\psi_L^* \end{pmatrix} = (\psi_M)' = \psi_M'$ qed.

$\psi_L = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow i\sigma_y\psi_L^* = \begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix} \rightarrow \psi_M = \begin{pmatrix} \psi_L \\ i\sigma_y\psi_L^* \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \beta^* \\ -\alpha^* \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}$

5) It has two independent complex components : α and β

6) $(i\gamma^\mu\partial_\mu - m)\psi_M = (i\gamma^\mu\partial_\mu - m)\begin{pmatrix} \psi_L \\ i\sigma_y\psi_L^* \end{pmatrix} = 0 \qquad$ with $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$

$\rightarrow \begin{cases} -m\psi_L - \sigma^\mu\sigma_y\partial_\mu\psi_L^* = 0 \\ i\bar\sigma^\mu\partial_\mu\psi_L - mi\sigma_y\psi_L^* = 0 \end{cases} \xrightarrow[\text{and take complex conjugation}]{\text{multiply by } i\sigma_y \text{ on left}} -mi\sigma_y\psi_L^* + i\underbrace{\sigma_y(\sigma^\mu)^*\sigma_y}_{\bar\sigma^\mu}\partial_\mu\psi_L = 0$

the two equations are the same.

7) $\psi_M' = e^{i\theta}\psi_M = \begin{pmatrix} e^{i\theta}\psi_L \\ e^{i\theta} i\sigma_y \psi_L^* \end{pmatrix}$ in the CR

$$(\psi_M')^c = -i\begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}\begin{pmatrix} e^{-i\theta}\psi_L^* \\ e^{-i\theta} i\sigma_y \psi_L \end{pmatrix} = -i\begin{pmatrix} e^{-i\theta} i\psi_L \\ -e^{-i\theta}\sigma_y\psi_L^* \end{pmatrix} = e^{-i\theta}\begin{pmatrix} \psi_L \\ i\sigma_y\psi_L^* \end{pmatrix} = e^{-i\theta}\psi_M$$

$\neq \psi_M'$ except if $\theta = 0$ or π.

Therefore $e^{i\theta}\psi_M$ is not a Majorana bi-spinor.

$$\mathcal{L} = \psi_L^+ i\bar{\sigma}^\mu\partial_\mu\psi_L - \frac{m}{2}\left[\psi_L^+(i\sigma_y\psi_L^*) + (i\sigma_y\psi_L^*)^+\psi_L\right] \xrightarrow[\psi_L \to e^{i\theta}\psi_L]{U(1)}$$

$$\psi_L^+ \cancel{e^{-i\theta}} i\bar{\sigma}^\mu \cancel{e^{i\theta}}\partial_\mu\psi_L - \frac{m}{2}\left[\psi_L^+\underbrace{e^{-i\theta}e^{-i\theta}}_{e^{-i2\theta}\neq 1}(i\sigma_y\psi_L^*) + (i\sigma_y\psi_L^*)^+\underbrace{e^{i\theta}e^{i\theta}}_{e^{i2\theta}\neq 1}\psi_L\right] \neq \mathcal{L}$$

$\mathcal{L}$ is not $U(1)$ invariant.

Majorana mass term : $\psi_L^+(i\sigma_y\psi_L^*) + (i\sigma_y\psi_L^*)^+\psi_L$ with $\psi_L = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

$$= (\alpha^* \ \beta^*)\begin{pmatrix}\beta^* \\ -\alpha^*\end{pmatrix} + (\beta \ -\alpha)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \alpha^*\beta^* - \beta^*\alpha^* + \beta\alpha - \alpha\beta$$

if $[\alpha,\beta] = 0$ then $\alpha\beta = \beta\alpha$ and $\alpha^*\beta^* = \beta^*\alpha^* \rightarrow$ Maj. mass term vanishes.

if $\{\alpha,\beta\} = 0$ then $\beta\alpha = -\alpha\beta$ and $\beta^*\alpha^* = -\alpha^*\beta^* \rightarrow$ ——— $= 2\alpha^*\beta^* + 2\beta\alpha \neq 0$.

8) A Majorana field does not have a global $U(1)$ invariance. We can not do the Weyl construction. We can not gauge the $U(1)$ symmetry. Hence there is no electric charge. The Majorana field does not couple to the electromagnetic field. The Dirac equation is $[i\gamma^\mu\partial_\mu - m]\psi_M(x) = 0$: the same as in the absence of electromagnetic field.

9) $(i\tilde{\gamma}^\mu)^* = i\tilde{\gamma}^\mu$ ie. $i\tilde{\gamma}^\mu$ is real $\Rightarrow (i\tilde{\gamma}^\mu\partial_\mu - m)$ is a real operator.

$\Rightarrow$ One can find a solution $\tilde{\psi}(x)$ or $(i\tilde{\gamma}^\mu\partial_\mu - m)\tilde{\psi}(x) = 0$ which is real, ie. $\tilde{\psi}(x)^* = \tilde{\psi}(x)$. Reality condition. Charge conjugation in the Majorana rep. would be $\tilde{\psi} \xrightarrow{c} \tilde{\psi}^c = \tilde{\psi}^*$ ie. just complex conjugation.

$$\tilde{\gamma}^5 = i\tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3 = \begin{pmatrix} -\sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}$$

$\tilde{\gamma}^5\tilde{\psi} = \pm\tilde{\psi}$ with $\tilde{\psi} = \begin{pmatrix}\alpha \\ \beta \\ \gamma \\ \delta\end{pmatrix}$ such that $\tilde{\psi}^* = \tilde{\psi}$ ie. $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

$$\Leftrightarrow \begin{cases} i\beta = \pm\alpha \\ -i\alpha = \pm\beta \\ -i\delta = \pm\gamma \\ i\gamma = \pm\delta \end{cases}$$

which is not possible with ———

Therefore a Majorana bispinor can not be a Weyl bispinor at the same time.

10)

	Dirac	Weyl	Majorana	neutrino
spin	1/2	1/2	1/2	1/2
charge	$-e$ or 0	$-e$ or 0	0	0
mass	m or 0	0	m or 0	finite small m
indepent complex components	4	2	2	?

Therefore a neutrino could be described either by a Dirac or by a Majorana field. But not by a Weyl field.

11) In the CR, $\psi_D = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{C}$

$$= \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \underbrace{\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}}_{\substack{\text{left} \\ \text{Weyl} \\ \text{bi-spinor}}} + \underbrace{\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}}_{\substack{\text{right} \\ \text{Weyl} \\ \text{bi-spinor}}} \quad \text{with } \psi_L = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \ \psi_R \equiv \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

In the Majorana representation, $\tilde{\psi}_D = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \in \mathbb{C}$

$$\tilde{\psi}_D = \begin{pmatrix} \alpha_R \\ \beta_R \\ \gamma_R \\ \delta_R \end{pmatrix} + i \begin{pmatrix} \alpha_I \\ \beta_I \\ \gamma_I \\ \delta_I \end{pmatrix} = \tilde{\psi}_M^{(R)} + i\,\tilde{\psi}_M^{(I)} \quad \text{where } \alpha_R \equiv \operatorname{Re}\alpha \in \mathbb{R}, \ \alpha_I \equiv \operatorname{Im}\alpha \in \mathbb{R}$$

and $\left(\tilde{\psi}_M^{(R)}\right)^* = \tilde{\psi}_M^{(R)}$ and $\left(\tilde{\psi}_M^{(I)}\right)^* = \tilde{\psi}_M^{(I)}$ are two different Majorana bi-spinors.

Exercise B: Massless Dirac equation in 1+1 spacetime.

1) $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\mu,\nu = 0,1 = t,x$ $\quad \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We need 2 anticommuting matrices: we can take 2 Pauli matrices. For example, $\gamma^0 = \sigma_x$ and $\gamma^1 = i\sigma_y$. γ^0 is off diagonal and exchanges the two components of a spinor: it acts as a parity matrix.

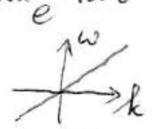
2) σ_z anticommutes with σ_x and σ_y; $\sigma_z^2 = \mathbb{1}$ and $\sigma_z^+ = \sigma_z$. Actually $\pm\sigma_z$ would work. But if $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ is a spinor with ψ_L, ψ_R two complex numbers, we would like to have $\gamma^5 \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = (-1)\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ and $\gamma^5 \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = (+1)\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$ and therefore choose $\gamma^5 = -\sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

3) $i\gamma^\mu\partial_\mu \psi = 0$ with $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow \begin{pmatrix} 0 & i\partial_t + i\partial_x \\ i\partial_t - i\partial_x & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$

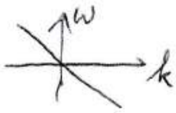
$$\begin{cases} (i\partial_t - i\partial_x)\psi_L = 0 & \text{i.e. } \bar{\sigma}^\mu = (1,-1) \\ (i\partial_t + i\partial_x)\psi_R = 0 & \text{i.e. } \sigma^\mu = (1,1) \end{cases} \Leftrightarrow \begin{cases} i\partial_t\psi_L = -(-i\partial_x)\psi_L \\ i\partial_t\psi_R = +(-i\partial_x)\psi_R \end{cases}$$

4) $i\partial_t \psi_R = -i\partial_x \psi_R$ with $\psi_R(t,x) = \psi_0\, e^{ikx} e^{-i\omega t}$ $\psi_0 \in \mathbb{C}$

$i(-i\omega)\psi_R = -i \times i k\, \psi_R \rightarrow \omega = +k$



$i\partial_t \psi_L = i\partial_x \psi_L \rightarrow \omega = -k$



5) $(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu)\psi = 0$ with $A^\mu = (0, -E_x t)$ and $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$

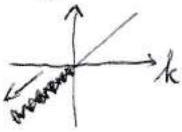
so that $i\partial_t\psi_R + i\partial_x\psi_R + eE_x t\,\psi_R = 0$

6) $\psi_R(t,x) = \psi_0\, e^{ikx} e^{-i\int^t dt'\,\omega(t')}$

$i.-i\omega(t)\psi_R(t,x) + i.i\,k\,\psi_R(t,x) + e E_x t\, \psi_R(t,x) = 0$

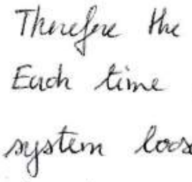
d'où $\omega(t) = k - eE_x t \rightarrow \dot{\omega}(t) = -eE_x$

7) The electric field is decreasing the energy of each right mover. Therefore the electric field is removing right movers.



Each time ω changes by $\frac{2\pi}{L}$ (as $\omega = k$ and $\Delta k = \frac{2\pi}{L}$ on a ring), the system looses 1 right mover. As $\dot{\omega} = -eE_x$, this happens every $\Delta t = \frac{2\pi/L}{eE_x}$.

Therefore $\frac{dN_R}{dt} = -\frac{eE_x}{2\pi/L}$ and $\frac{dn_R}{dt} = -\frac{eE_x}{2\pi}$.

8)  as $\dot{\omega}(t) = +eE_x$ because $(i\partial_t - i\partial_x - eE_x t)\psi_L(t,x) = 0$.

Therefore $\frac{dn_L}{dt} = +\frac{eE_x}{2\pi} \Rightarrow \frac{d}{dt}(n_R + n_L) = 0$ and $\frac{d}{dt}(n_R - n_L) = -\frac{eE_x}{\pi}$.

The total number of Weyl particles is conserved but not the difference between left and right movers.

9) For the massless Dirac Lagrangian in 1+1 $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu)\psi$ is invariant under $\psi \rightarrow e^{i\theta}\psi$ global vector $U(1)$ i.e. $\psi \rightarrow e^{i\theta}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ in CR

and under $\psi \rightarrow e^{i\theta\gamma^5}\psi$ global axial $U(1)$ ie $\psi \rightarrow \begin{pmatrix}e^{-i\theta}\psi_L \\ e^{i\theta}\psi_R\end{pmatrix}$ in CR

From Noether's theorem, there are two conserved currents $J_V^\mu = \bar{\psi}\gamma^\mu\psi$ and $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$.

The conserved charges are $J_V^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \psi_L^*\psi_L + \psi_R^*\psi_R = n_L + n_R$

and $J_A^0 = \bar{\psi}\gamma^0\gamma^5\psi = \psi^\dagger\gamma^5\psi = (\psi_L^* \ \psi_R^*)\begin{pmatrix}-\psi_L \\ \psi_R\end{pmatrix} = \psi_R^*\psi_R - \psi_L^*\psi_L$

$= n_R - n_L$

Conservation of these charges means that $\frac{d}{dt}(N_L + N_R) = 0$ and $\frac{d}{dt}(N_R - N_L) = 0$. At least classically. Indeed

$$Q_V = \int dx\, J_V^0 = \int dx(n_L + n_R) = N_R + N_L$$

$$Q_A = \int dx\, J_A^0 = \int dx(n_R - n_L) = N_R - N_L$$

10) $H|\psi\rangle = \mathcal{E}|\psi\rangle$

$\langle i|H|\psi\rangle = \mathcal{E}\psi_i$

$\Leftrightarrow \; -J\sum_j \left(\delta_{j,i-1}\,\psi_j + \delta_{j,i}\,\psi_{j+1}\right) = \mathcal{E}\psi_i$

$\Leftrightarrow \; \mathcal{E}\psi_i = -J(\psi_{i-1} + \psi_{i+1})$

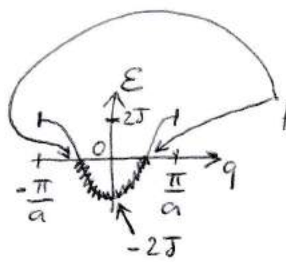
ansatz : $\psi_j = \psi_0\, e^{iqaj}$ $\qquad \psi_{j\pm1} = e^{\pm iqa}\,\psi_j$

$\Rightarrow \; \mathcal{E}\psi_j = -J\left(e^{-iqa} + e^{iqa}\right)\psi_j \;\rightarrow\; \mathcal{E}(q) = -2J\cos(qa)$

$|\psi\rangle = |q\rangle$ such that $\psi_j = \langle j|q\rangle = e^{iqaj}$ (Bloch wave)

$-\pi \leq qa < \pi$ as $|q + \frac{2\pi}{a}\rangle$ and $|q\rangle$ are the same state

$\left(\langle j|q + \frac{2\pi}{a}\rangle = e^{iqaj}\,e^{i2\pi j} = e^{iqaj} = \langle j|q\rangle\right)$



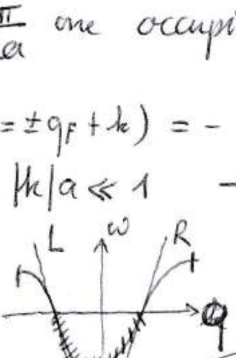
Fermi surface
(two points)

11) On a ring of length L, $\Delta q = \frac{2\pi}{L}$ and there are $\frac{2\pi/a}{2\pi/L} = \frac{L}{a}$ states in the first Brillouin zone (as many states as atoms in the chain on the ring). If there are $N = \frac{L}{a}$ atoms and $\frac{N}{2}$ electrons then at zero temperature, all states with $|q| \leq q_F$, $q_F = \frac{\pi}{2a}$ are occupied and $\mathcal{E}_F = \mathcal{E}(\pm q_F) = -2J\cos(\pm q_F a) = 0$.

$\mathcal{E}(q = \pm q_F + k) = -2J\cos\left(\pm\frac{\pi}{2} + ka\right) = -2J\cos\left(\frac{\pi}{2} \pm ka\right) = \pm 2J\sin(ka) \simeq \pm 2Jka$

when $|k|a \ll 1 \;\rightarrow\; \mathcal{E}(k) \simeq \pm 2Jka = \pm \hbar v k$ with $v \equiv \frac{2Ja}{\hbar}$

12)



$+\hbar v k$ near $+q_F$

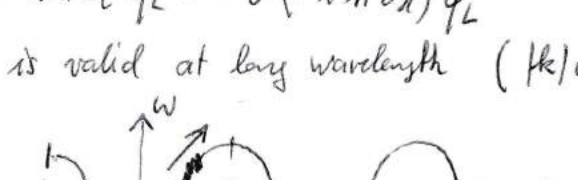
$-\hbar v k$ near $-q_F$

R: $i\hbar\partial_t\psi_R = v(-i\hbar\partial_x)\psi_R$

L: $i\hbar\partial_t\psi_L = -v(-i\hbar\partial_x)\psi_L$

This is valid at long wavelength ($|k|a \ll 1$) and low energy ($|\mathcal{E}| \ll J$).

13)



with an electric field $E_x < 0$

$\vec{E} = E_x \vec{u_x}$

This is just Bloch oscillations. The whole Fermi sea is moving as a block under the applied electric field. The "chiral anomaly" is just the transfer of left movers into right movers (and later in the opposite way) under an applied $\vec{E}$.