Exam for "Symmetries and quantum field theory"
Master 2 Concepts fondamentaux de la physique, parcours physique quantique
Teachers : J.N. Fuchs and G. Orso
December 18th 2015 from 9 : 00 till 12 : 00

Duration : 3h. Handwritten notes, typed lecture notes and typed sheets given during the lectures/exercise sessions are the only documents allowed (no books, computers, phones, etc.).

Problem : abelian Higgs model in 2+1 dimensions and vortices

In all of the following, we consider a flat $2+1$ dimensional spacetime and take units such that $c=1$ and $\hbar=1$.
For later use, we recall some vectorial operations in polar coordinates $\vec{x}=(x,y)=(r\cos\theta, r\sin\theta)$. We consider a function $f(\vec{x})=f(r,\theta)$ and a vector $\vec{V}(r,\theta)=V_r\vec{u}_r+V_\theta\vec{u}_\theta$ where $\vec{u}_r=\cos\theta\vec{u}_x+\sin\theta\vec{u}_y$ and $\vec{u}_\theta=-\sin\theta\vec{u}_x+\cos\theta\vec{u}_y$. The gradient is given by $\vec{\nabla}f=\vec{u}_r\partial_r f+\vec{u}_\theta\frac{1}{r}\partial_\theta f$ and the Laplacian by $\vec{\nabla}^2 f=\frac{1}{r}\partial_r(r\partial_r f)+\frac{1}{r^2}\partial_\theta^2 f$. The divergence is $\vec{\nabla}\cdot\vec{V}=\frac{1}{r}\partial_r(rV_r)+\frac{1}{r}\partial_\theta V_\theta$ and the (2D) curl is $(\vec{\nabla}\times\vec{V})_z=\frac{1}{r}\partial_r(rV_\theta)-\frac{1}{r}\partial_\theta V_r$.

1 Homogeneous vacuum state of the Goldstone model

Consider the Goldstone model for a complex scalar field $\phi(x^\mu)=\phi(t,x,y)$ with $\mu=0,1,2$. The Lagrangian density is

$$\mathcal{L}_G=(\partial_\mu\phi)^*(\partial^\mu\phi)-V(|\phi|)=(\partial_\mu\phi)^*(\partial^\mu\phi)-\lambda[|\phi|^2-\chi_0^2]^2 \quad (1)$$

with $\lambda>0$. The constant $\chi_0^2>0$ in the symmetry-broken phase and $\chi_0^2<0$ in the unbroken phase.
1. Obtain the corresponding Hamiltonian. Draw the potential V as a function of the real and imaginary parts of ϕ when $\chi_0^2>0$.

2. Find the groundstate and its energy when $\chi_0^2>0$. Is the groundstate unique ? If not, what is its degeneracy ? What are the low energy excitations ?

2 Homogeneous vacuum state of the abelian Higgs model

We now couple the complex field ϕ to a gauge field to obtain the Lagrangian density

$$\mathcal{L}_H=(D_\mu\phi)^*(D^\mu\phi)-V(|\phi|)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (2)$$

with $D_\mu\equiv\partial_\mu+iqA_\mu$ where q is the electric charge of the scalar field and the field strength $F_{\mu\nu}$ has the usual expression in terms of the vector potential A_μ.

3. Write the field strength $F_{\mu\nu}$ as a function of the electric and magnetic fields for this 2+1 spacetime. For later use, give the Lagrangian of the Maxwell field (alone) in terms of the electric and magnetic fields.

4. What are the low energy excitations above a symmetry-broken groundstate ($\chi_0^2>0$) ? Obtain their dispersion relation and their mass.

5. Explain the counting of the number of modes. How many modes in the symmetry-broken state ($\chi_0^2>0$) and how many modes in the unbroken state ($\chi_0^2<0$) ?

3 Inhomogeneous "vacuum state" of the Goldstone model : vortex

We come back to the Goldstone model of section (1) and study another type of "vacuum state" when $\chi_0^2>0$. It is not the state with the absolute minimum of energy but it is still quite stable. It is an **inhomogeneous** state such that

$$\phi(\vec{x})\approx\chi_0 e^{in\theta(\vec{x})} \quad (3)$$

where $\vec{x} = (r\cos\theta, r\sin\theta)$ in polar coordinates (r and θ). Equation (3) is only valid when r is sufficiently large (compared to a length scale r_c that will be discussed below) and does not describe what happens near $r \to 0$. For the moment, we can consider that $\phi(\vec{x}) \approx 0$ when $r < r_c$.

6. What is the condition on n for the above state to be single-valued ?

7. Show that for a static field, the Hamiltonian and Lagrangian densities are related by $\mathcal{H} = -\mathcal{L}$.

8. Compute the energy density for the above inhomogeneous "vacuum state". Then obtain the difference in total energy between the inhomogeneous and homogeneous vacuum states and show that it diverges with the size of space.

4 Inhomogeneous "vacuum state" of the abelian Higgs model : magnetic vortex

Consider the abelian Higgs model of section (2) in the presence of the above inhomogeneous "vacuum state" for the scalar field $\phi(\vec{x}) \approx \chi_0 e^{in\theta(\vec{x})}$ (still when $\chi_0^2 > 0$). We choose the vector potential to be of the form

$$A^\mu(\vec{x}) = (A^0, \vec{A}) \approx (0, \frac{n}{q}\vec{\nabla}\theta) \tag{4}$$

when $r \gg r_c$. For the moment we only consider the behavior when $r \gg r_c$ and will later study what happens when $r \ll r_c$.

9. Show that this choice ensures that the covariant derivative $D_\mu\phi$ vanishes when $r \gg r_c$. Show that the electric field vanishes everywhere in space.

10. Using question 7, obtain the energy density for a static field configuration as a function of ϕ and $F_{\mu\nu}$. Show that the total energy above the absolute (homogeneous) vacuum state is now finite.

11. Show that the above vector potential is a pure gauge and therefore corresponds to a vanishing magnetic field away from the origin. Compute the magnetic flux Φ across a disk of radius $r \gg r_c$. Is this flux quantized in units of the flux quantum $\Phi_0 \equiv \frac{2\pi\hbar}{|q|} = \frac{2\pi}{|q|}$? Does the magnetic field vanish everywhere in space ?

5 Nielsen and Olesen vortex solution

Here we continue to study the above inhomogeneous "vacuum state" of the abelian Higgs model when $\chi_0^2 > 0$.

12. Write the Euler-Lagrange equations of motion for the two fields ϕ and A^μ. Then simplify them in the static limit and insert the ansatz $\phi(r,\theta) = \chi(r)e^{in\theta}$ and $\vec{A}(r,\theta) = A_\theta(r)\vec{u}_\theta$ to obtain [1] :

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi}{dr}\right) = \chi[2\lambda(\chi^2 - \chi_0^2) + (qA_\theta - \frac{n}{r})^2] \tag{5}$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(rA_\theta)}{dr}\right) = 2q\chi^2(qA_\theta - \frac{n}{r}) \tag{6}$$

The boundary conditions are $\chi(r \to 0) = 0$, $\chi(r \to \infty) = \chi_0$, $A_\theta(r \to 0) = 0$ and $A_\theta(r \to \infty) = \frac{n}{qr}$.

These coupled equations were studied by Nielsen and Olesen in 1973, who found an approximate solution. We will solve them even more crudely by decoupling them in the limit $r \to \infty$. The aim is to understand qualitatively the behavior of the solution.

13. In equation (6), we assume that $\chi \approx \chi_0$ when r is large and we define $\delta A \equiv A_\theta - \frac{n}{qr}$. Show that it satisfies

$$\frac{d^2\delta A}{dr^2} \approx 2q^2\chi_0^2\delta A \tag{7}$$

1. For the second equation, we give the following hints. First show that in the static limit $\partial_\mu F^{\mu\nu}$ becomes $\vec{\nabla} \times \vec{B}$. Then use $\vec{A} = A_\theta(r)\vec{u}_\theta$ to show that $\vec{B} = B(r)\vec{u}_z = \frac{1}{r}\frac{d(rA_\theta)}{dr}\vec{u}_z$. Eventually, use the fact that in cylindrical coordinates $\vec{\nabla} \times \vec{B} = -\frac{dB}{dr}\vec{u}_\theta$.

Solve this equation (up to an undetermined normalization constant) using the boundary condition at $r \to \infty$ to find δA. Then obtain the corresponding magnetic field $B(r)$ when r is large. Assume that this solution remains valid when $r \to 0$ to fix the normalization constant. Sketch B as a function of r. What is its characteristic decay length scale l_B ?

14. In equation (5), we assume that $A_\theta \approx \frac{n}{qr}$ and that $\delta\chi \equiv \chi - \chi_0$ is small compared to χ_0 when r is large. Show that the latter satisfies :

$$\frac{d^2\delta\chi}{dr^2} \approx 4\lambda\chi_0^2\delta\chi \tag{8}$$

Solve this equation using the boundary conditions at $r \to \infty$ and at $r \to 0$ (by assuming that the solution obtained at large r remains valid when $r \to 0$). Sketch χ as a function of r. What is its characteristic length scale l_χ ? Relate the two length scales l_B and l_χ to the masses of the excitations obtained in question 4.

6 Superconductivity and Abrikosov vortex

The abelian Higgs model in the static limit is actually identical to the Landau-Ginzburg (LG) theory of superconductivity. The LG energy functional in 2 space dimensions is

$$E[\phi(\vec{x}), \vec{A}(\vec{x})] = \int dxdy \left\{ |(-\vec{\nabla} + iq\vec{A})\phi|^2 + V(|\phi|) + \frac{B^2}{2} \right\} \tag{9}$$

with $V(|\phi|) = \lambda[|\phi|^2 - \chi_0^2]^2$. In the energy functional, the scalar field $\phi(\vec{x})$ is the condensate wavefunction describing Cooper pairs of electrons with charge $q = -2e < 0$ (also called the order parameter of the superconducting transition). The quantity $\chi_0^2 \propto T_c - T$ changes sign at the critical temperature T_c (χ_0^2 is negative in the high temperature normal phase and positive in the low temperature superconducting phase). The inhomogeneous "vacuum state" that we have discussed above actually corresponds to having a magnetic vortex in a 2D superconductor as discovered by Abrikosov.

15. In this condensed matter context, what is the physical meaning of the two length scales l_B and l_χ obtained above ? To what famous effect is the length scale l_B related to ? What is the importance of the ratio l_B/l_χ ?

16. Give an argument for the stability of the inhomogeneous "vacuum state" with a single vortex. Why does it not decay rapidly (via a quantum transition for example) to the absolute homogeneous groundstate ?

References :

L.H. Ryder, *Quantum field theory* (Cambridge university press, 1st edition, 1985), pages 406-413.
H.B. Nielsen and P. Olesen, Nuclear Physics B **61**, 45 (1973).