Exam for "Symmetries and quantum field theory"

Master 2 Concepts fondamentaux de la physique, parcours physique quantique

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Duration : 3h. Handwritten notes, typed lecture notes and typed sheets given during the lectures/exercise sessions are the only documents allowed (no books, computers, phones, etc.). The two problems are independent. One complete problem should already earn you the maximum grade.

1 Non-relativistic limit of a scalar field theory

Consider a complex relativistic and scalar field ϕ in $3+1$ spacetime. The Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - m^2 \phi^* \phi - \lambda(\phi^*\phi)^2 \tag{1}$$

where $\lambda \geq 0$ and $m > 0$ and we have set the light velocity $c = 1$ and $\hbar = 1$.

1) Show that there is a $U(1)$ symmetry.

2) Obtain the Euler-Lagrange equation of motion for the ϕ field. What is the conjugate field of ϕ ? In the particular case $\lambda = 0$, what is the corresponding dispersion relation ? Plot it. Expand the positive branch of the dispersion relation (still in the $\lambda = 0$ case) to second order in the wavevector (non-relativistic limit).

In the following, we will be interested in the non-relativistic limit of the above field theory. In this limit, we separate fast and slow temporal oscillations of the field by setting

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2m}} e^{-imt} \varphi(t, \vec{x}) \tag{2}$$

which defines the slow field (or enveloppe) φ. The factor $\frac{1}{\sqrt{2m}}$ is just a normalization choice.

3) Starting from the equation of motion for ϕ (see question 2), take the non-relativistic limit to obtain an equation of motion for the slow field φ. In the limit $\lambda = 0$, this should be a familiar equation in another context. Check that the corresponding dispersion relation agrees with that found in question 2. Plot this dispersion relation and compare it to the relativistic case. What can you say about the number of modes ?

4) Non-relativistic limit of the Lagrangian. Write the Lagrangian for the slow field. Show that

$$\mathcal{L} \approx i\varphi^* \partial_t \varphi - \frac{1}{2m} \nabla\varphi^* \cdot \nabla\varphi - \frac{g}{2}|\varphi|^4 \tag{3}$$

and give the expression of the coupling constant g in terms of λ and m. From the Euler-Lagrange equation, recover the equation of motion for the slow field.

5) What is the conjugate field of φ ? Write the equal time commutation relations for the slow field and its conjugate field. In the following, we stick to the classical field theory (except at the end of question 10).

6) Obtain the Hamiltonian (not density) for the slow field.

7) Recall the Noether current J^μ corresponding to the $U(1)$ symmetry of the relativistic field theory. Obtain J^0 and $\vec{J}$ in the non-relativistic limit and write the local conservation law. What is the physical meaning of J^0 ? And therefore of the conserved charge $Q = \int d^3x J^0$? What is the main difference for Q with respect to the relativistic case ?

8) In condensed matter physics, the vacuum is not empty of particles but rather corresponds to a finite density of particles. What we actually have in mind is a gas of bosonic atoms such as that produced in atomic physics labs (the field known as "cold atoms"). In order to impose this finite density, we add to the Hamiltonian a Lagrange multiplier known as the chemical potential μ. The Lagrangian now reads :

$$\mathcal{L} = i\varphi^* \partial_t \varphi - \frac{1}{2m} \nabla\varphi^* \cdot \nabla\varphi + \mu|\varphi|^2 - \frac{g}{2}|\varphi|^4 \tag{4}$$

The above field theory depends on three parameters : m, g and μ. In the following, $m > 0$ and $g > 0$ are fixed and we tune μ (in practice, in a gas of bosonic atoms, μ is controlled by the density or by the temperature T. For example, at fixed density, $\mu \to -\infty$ when $T \to \infty$ and $0 < \mu < \infty$ when $T \to 0$).

We wish to find the classical groundstate of the Hamiltonian corresponding to the above Lagrangian. What is the energy potential $V(\varphi^*, \varphi)$? When $\mu < 0$, find a groundstate and discuss its stability and degeneracy. Same questions when $\mu > 0$. What happens at $\mu = 0$?

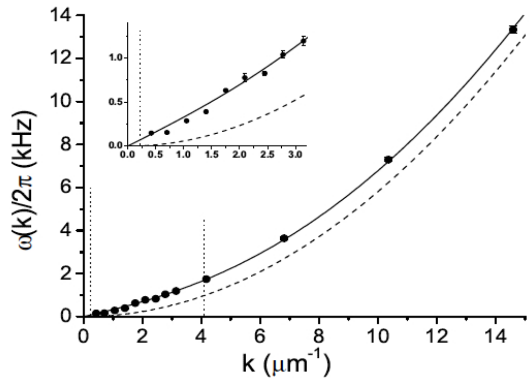

FIGURE 1 – Dispersion relation of the elementary excitations in a Bose-Einstein condensate of rubidium atoms. The inset shows a zoom of the low frequency and long wavelength region. Taken from J. Steinhauer et al., Phys. Rev. Lett. **88**, 120407 (2002)

9) We now change variables and represent the slow field in density/phase representation $\varphi(x) = \sqrt{n(x)}e^{i\theta(x)}$. Show that the above Lagrangian becomes (up to a total derivative)

$$\mathcal{L} = (\mu - \partial_t \theta)n - \frac{n}{2m}(\nabla\theta)^2 - \frac{(\nabla n)^2}{8mn} - \frac{gn^2}{2} \tag{5}$$

What is the conjugate field of the phase field θ ? Write the corresponding equal-time commutation relations. Show that the number operator $N(t) = \int d^3x\, n(t, \vec{x})$ has a simple equal-time commutator with the phase field $\theta(t, \vec{x})$.

10) We assume $\mu > 0$ and consider a classical groundstate $\varphi_0 = \sqrt{\frac{\mu}{g}}e^{i0} = \sqrt{\frac{\mu}{g}}$. We introduce the notation $\bar{n} \equiv \frac{\mu}{g}$. Quadratize the Lagrangian close to this groundstate by setting $\varphi(x) = \sqrt{n(x)}e^{i\theta(x)}$ with $n(x) = \bar{n} + \rho(x)$ such that $|\rho| \ll \bar{n}$ and $|\theta| \ll 1$. Up to a constant and a total derivative, show that

$$\mathcal{L} \approx -\rho\partial_t\theta - \frac{\bar{n}}{2m}(\nabla\theta)^2 - \frac{(\nabla\rho)^2}{8m\bar{n}} - \frac{g\rho^2}{2} \tag{6}$$

Compare the structure of the above quadratized Lagrangian with that of the relativistic Goldstone model that we studied in class. What are the main differences ?

11) Obtain the equation of motion for θ and for ρ. These are coupled equations. Combining them, obtain an equation of motion for the density fluctuation ρ alone [1].

12) Obtain the corresponding dispersion relation. Compare the cases $g = 0$ and $g > 0$. In the long wavelength limit, obtain the velocity of these excitations. What are the physical nature and usual name of this collective mode ? The qualitative change in the long wavelength dispersion relation upon turning on a finite positive g is responsible for the phenomenon of superfluidity.

13) In the figure, we reproduce a measurement of the dispersion relation for the collective mode in a Bose-Einstein condensate of ^{87}Rb atoms. Estimate the order of magnitude of the velocity for the excitations from the formula you obtained (is it meters/second or picometers/second or kilometers/picosecond or etc ?). The gas contains 10^5 atoms and has the shape of a cigar with axial radius 28 microns and radial radius 3 microns. The mass of a nucleon is 1.7×10^{-27} kg (how many nucleons in ^{87}Rb ?) and the interaction constant $g = \frac{4\pi\hbar^2}{m}a$ where $a \approx 5$ nm is the scattering length and $\hbar = 1.05 \times 10^{-34}$ J.s. Does this order of magnitude agrees with the experimental result ?

14) In the light of the Goldstone theorem, what can you say of the above excitations ? Why is there no massive Higgs mode ?
Indication : use the answer to question 3.

References :

A. Zee, *Quantum field theory in a nutshell* (Princeton university press, 1st edition, 2003), pages 172-175 and 257-260.

1. If instead, you eliminate ρ, you should obtain the same wave equation for the phase oscillation θ alone.

2 Vector field theories in 2+1 dimensions

In this problem, we consider different field theories for a vector field $A^\mu = (A^0, A^i) = (A_0, \vec{A})$ in 2+1 spacetime dimension, where $\mu = t, x, y = 0, 1, 2$ and $i = 1, 2$. The metric is $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1)$ and the field strength is $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

2.1 Maxwell

We first consider the Maxwell theory for a gauge field in 2+1. The Maxwell Lagrangian is

$$\mathcal{L}_M = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu \tag{7}$$

where J^μ is a conserved current (or a source).

1) Derive the Euler-Lagrange equation of motion for A^μ. Show that a consequence is that the current is conserved.

2) Show that the action is gauge invariant.

3) Write the magnetic B and electric $\vec{E} = (E_x, E_y)$ fields as a function of $A^\mu = (A^0, A^1, A^2) = (V, A_x, A_y)$. Write the field strength in matrix form as a function of the magnetic and electric fields. In 2+1, the parity transformation P acts as $x^\mu = (x^0, x^1, x^2) \to (x^0, -x^1, x^2)$ (and NOT as $x^\mu = (x^0, x^1, x^2) \to (x^0, -x^1, -x^2)$ as parity should have $\det P = -1$). Show that the magnetic field is a pseudo-scalar and the electric field is a true vector.

4) From the definition of the field strength and from the EL equation of motion, obtain all the Maxwell equations in 2+1. Compare with the 3+1 case.

5) From the equation of motion for the A^μ field (when $J^\mu = 0$), obtain the dispersion relation. How many modes are there ?

Indication : consider the radiation gauge.

2.2 Proca

Consider now the Proca Lagrangian for a vector field A^μ

$$\mathcal{L}_P = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m_A^2}{2} A_\mu A^\mu \tag{8}$$

in the absence of a source.

6) Is the corresponding action gauge-invariant ?

7) Obtain the EL equation of motion. Is it gauge invariant ? Then obtain the equation of motion for the A^μ field only by showing that $\partial_\mu A^\mu = 0$ when $m_A \neq 0$. Conclude on the possibility of having a massive gauge field.

8) Obtain the corresponding dispersion relation. How many modes are there ?

Indication : go in the rest frame.

2.3 Chern-Simons

We now consider an alternative theory for a gauge field A^μ in 2+1. The Chern-Simons (CS) Lagrangian is

$$\mathcal{L}_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu \tag{9}$$

where $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric symbol with $\epsilon^{012} = +1$, $\epsilon_{\mu\nu\rho} = \epsilon^{\mu\nu\rho}$ (can you justify this last relation ?) and κ is a real parameter. The current is assumed to be conserved.

9) Show that the CS action is gauge invariant (provided what ?). Show that the CS term $\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ is not invariant under the parity transformation as defined in question 3.

10) Carefully derive the EL equation of motion and show that it is gauge invariant. Show that in the absence of a source J^μ (called a pure CS theory), the field strength vanishes and A^μ is a pure gauge.

11) Still in absence of a source, and in a gauge such that $A_0 = 0$, show that the Hamiltonian $H = 0$. Comment on the interest of such a pure CS theory.

2.4 Maxwell-Chern-Simons

Combining the two previous theories, we obtain the Maxwell-Chern-Simons Lagrangian

$$\mathcal{L}_{MCS} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{10}$$

12) Is the corresponding action gauge-invariant ?

13) Carefully obtain the EL equation of motion and express it in terms of the field strength only.

14) We introduce the dual field strength $\tilde{F}^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$. Obtain $F^{\mu\nu}$ as a function of its dual. Show that $\partial_\mu\tilde{F}^\mu = 0$. Express $\tilde{F}^\mu$ in terms of the magnetic and electric fields. To which Maxwell equation is $\partial_\mu\tilde{F}^\mu = 0$ equivalent to ?
Indication : use the identity $\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha\delta^\rho_\beta - \delta^\nu_\beta\delta^\rho_\alpha$.

15) Rewrite the equation of motion of question 13 in terms of the dual field strength only. From there, show that

$$(\Box + \kappa^2)\tilde{F}^\mu = 0 \tag{11}$$

and give the corresponding dispersion relation.
Indication : consider $\kappa^2\tilde{F}^\mu$ and use the identity $\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha\delta^\rho_\beta - \delta^\nu_\beta\delta^\rho_\alpha$.

2.5 Conclusion

16) Is it possible to obtain a massive gauge theory in 2+1 ? And in 3+1 ? Compare different ways of having a mass for a gauge field. And for a vector field.

References :

A. Zee, *Quantum field theory in a nutshell* (Princeton university press, 1st edition, 2003), sections VI.1 to VI.3.
G.V. Dunne, *Aspects of Chern-Simons theory*, arXiv :hep-th/9902115.