Exam for "Symmetries and quantum field theory"

Master 2 Concepts fondamentaux de la physique, parcours physique quantique

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Duration : 3h.

Handwritten notes and typed sheets given during the lectures/exercise sessions are the only documents allowed. The two problems are independent. We suggest to spend about 1.5h on each problem.

1 Axion electrodynamics for 3+1 topological insulators

We consider a 3D insulating (i.e. which does not conduct electricity) material featuring a magneto-electric effect. The latter is an unusual effect : for example, an electric field induces a magnetic polarization and a magnetic field produces an electric polarization. The electrodynamics of such an insulator can be shown to be described by the following action [1] :

$$S = S_0 + S_\theta = \int_M d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha}{2\pi} \epsilon^{\mu\nu\rho\sigma} \theta (\partial_\mu A_\nu)(\partial_\rho A_\sigma) \right)$$

where M is the space-time manifold, $\alpha = e^2/(4\pi\hbar)$ is the fine structure constant and $\epsilon^{\mu\nu\rho\sigma} \equiv \epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric symbol such that $\epsilon^{0123} = +1$ (we remind that it has the same expression in any frame). The first term is the Maxwell action and depends on the gauge field A^μ. The second is the θ-term and depends on a real field $\theta(x)$ called the axion field. This field varies slowly in space-time and plays the role of an order parameter field : it has no dynamics on its own (there is no kinetic energy for this field) and may be thought of as an external field.

We first study the nature of the θ-term **when** $\theta =$ **constant** (as relevant for a bulk homogeneous crystal).

1. Show that S_0 is a Lorentz scalar. Show that S_θ is a Lorentz pseudo-scalar. The following identity, valid for a 4×4 matrix C with elements $C^\mu{}_\nu$, may be useful : $\epsilon_{\mu\nu\rho\sigma} C^\mu{}_\alpha C^\nu{}_\beta C^\rho{}_\gamma C^\sigma{}_\delta = \epsilon_{\alpha\beta\gamma\delta} \det(C)$.

2. Write the action S as a function of the electric $\vec{E}$ and magnetic $\vec{B}$ fields. Is S_θ gauge invariant ? Write S_θ as a function of the electromagnetic field strength $F^{\mu\nu}$ and its dual $\widetilde{F}^{\mu\nu}$.

3. Obtain the Euler-Lagrange equation for the A_μ field. Write the corresponding Maxwell equations for $\vec{E}$ and $\vec{B}$. What is the effect of the θ term ?

4. Starting from the expression of S_θ in terms of the gauge field A_μ show that S_θ can be written as a total derivative. Use this property to express S_θ as an integral over the manifold boundary ∂M. In hindsight, explain the results of 3).

We admit that θ is actually an angular variable and therefore defined modulo 2π. We take $\theta \in [0, 2\pi[$ in the following. The properties obtained in 3) and 4) mean that S_θ is a topological term, i.e. it only depends on the topology of the field A^μ (the way the field wraps around the space-time manifold) and not on its precise configuration.

5. What is the behavior of $\vec{E} \cdot \vec{B}$ under a parity transformation P ? Under a time-reversal transformation T ? Under PT ? From this considerations, deduce the behavior of S_θ and of S_0 under P, T and PT.

6. Assume that the action has time reversal symmetry (TRS). What are the possible values of θ ? A trivial insulator corresponds to $\theta = 0$. Knowing that a topological insulator can be defined as an insulating crystal with TRS and which is distinct from a trivial insulator, what is its θ value ?
What are the possible θ values for a 3D parity invariant insulator ? And for a 3D insulator without P and T symmetries ?

We now take a general situation **in which** $\theta \neq$ **constant** as relevant for an inhomogeneous crystal, e.g. (remember that θ has no dynamics and can be thought of as a given external field).

7. Obtain the Euler-Lagrange equation for the A_μ field and give the corresponding equations for $\vec{E}$ and $\vec{B}$ when $\theta \neq$ constant. What is the effect of the axion field ? Does it affect all four Maxwell equations ? These modified Maxwell equations are known as "axion electrodynamics".

1. Starting from S.I. units, we take $c \equiv 1$ and $\epsilon_0 \equiv 1$ (hence $\mu_0 = 1$) so that the fine structure constant $\alpha \equiv e^2/(4\pi\epsilon_0\hbar c) = e^2/(2h) \approx 1/137$ and the conductance quantum $e^2/h = 2\alpha$.

8. Compute the (topological part of the) electrical current defined by $J^\mu = -\frac{\delta S_\theta}{\delta A_\mu}$ and show that it vanishes if $\theta =$ constant.

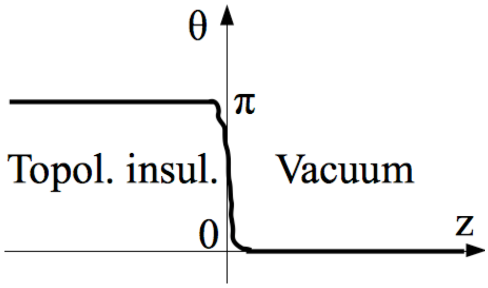

FIGURE 1 – Domain wall in the axion field $\theta(z)$ at the surface of a topological insulator.

9. Consider the surface (at $z = 0$) of a semi-infinite 3D topological insulator ($z < 0$ is the topological insulator and $z > 0$ is the vacuum which is assumed to be a trivial insulator). This is represented as a domain wall in the field $\theta(t, x, y, z) = \theta(z)$, see figure 1. Next apply an electric field $\vec{E} = E_y \vec{e}_y$ (with $E_y(x, y, z, t) =$ constant and $\vec{B} = 0$) and compute the 2D current density defined by $\vec{j}(x, y) = \int dz \vec{J}(x, y, z)$. Obtain the Hall conductivity σ_{xy} defined by $j_x = \sigma_{xy} E_y$. Show that – in the absence of a magnetic field – there is a Hall effect and give its value in units of the conductance quantum $e^2/h = 2\alpha$.

10. What determines the direction (along $\vec{e}_x$ or $-\vec{e}_x$) of the Hall current ? Does this make sense ?

References :

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X.L. Qi, T.L. Hugues and S.C. Zhang, Phys. Rev. B **78**, 195424 (2008).
A.M. Essin, J.E. Moore and D. Vanderbilt, Phys. Rev. Lett. **102**, 146805 (2009).

2 Boron nitride and the 2+1 Dirac equation

We consider the low energy properties of conduction electrons in boron nitride (BN). This 3D material is made of stacked 2D planes. Each plane is a honeycomb lattice (see figure 2(a)), with nearest neighbor distance $a \sim 1$Å. In the following, we study a single layer of BN and neglect the spin of the electrons. In a tight-binding model, each atom (either B or N) may accommodate a single conduction electron. The latter can hop from an atom to a nearest neighbor atom with a probability amplitude $t \sim 1$ eV. The on-site energy ϵ_α for an electron depends on the atom type ($\alpha = N$ or B). We choose the zero of energy such that $\epsilon_N = -\epsilon_B > 0$. We call $c^\dagger_{i\alpha}$ the creation operator of an electron on the atom (i, α), where i indicates a triangular Bravais lattice site and α which of the two atoms in the unit cell (see figure 2(a)). The Hamiltonian is given by :

$$H = \sum_{i,\alpha} \epsilon_\alpha c^\dagger_{i\alpha} c_{i\alpha} - t \sum_{\langle i\alpha, j\alpha' \rangle} \left(c^\dagger_{i\alpha} c_{j\alpha'} + \text{H.c.} \right) \tag{1}$$

where the notation $\langle i\alpha, j\alpha' \rangle$ indicates that the sum is over every pair of nearest neighbor atoms in the lattice.

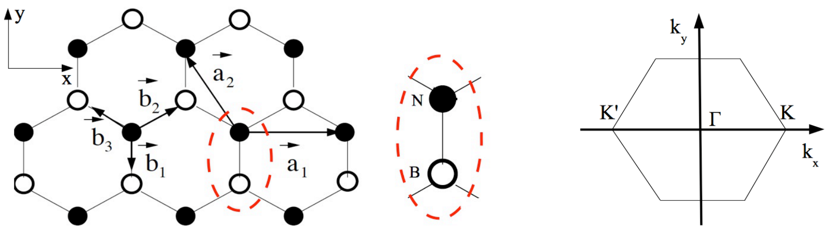
1. Performing a Fourier transform on the Bravais lattice index, show that the Hamiltonian may be written as :

$$H = \sum_{\vec{k} \in \text{BZ}} (c^\dagger_{\vec{k}N}, c^\dagger_{\vec{k}B}) \cdot \begin{pmatrix} \Delta & f(\vec{k})^* \\ f(\vec{k}) & -\Delta \end{pmatrix} \cdot \begin{pmatrix} c_{\vec{k}N} \\ c_{\vec{k}B} \end{pmatrix} = \sum_{\vec{k} \in \text{BZ}} \Psi^\dagger_{\vec{k}} \cdot H(\vec{k}) \cdot \Psi_{\vec{k}} \tag{2}$$

where BZ indicates the Brillouin zone (see figure 2(b)) and $\Psi_{\vec{k}}$ is a two-component object for each $\vec{k}$. Give the expression of the function $f(\vec{k})$ and of Δ.

2. What are the eigen-energies $\epsilon^\pm_{\vec{k}} = \pm\epsilon_{\vec{k}}$ (with $\epsilon_{\vec{k}} > 0$) of the single electron states of (2) ?

3. Show that $\epsilon_{\vec{k}}$ has a minimum at two inequivalent values of $\vec{k}$ on the edges of the BZ : $\vec{k} = \pm\vec{K} = \pm(\frac{4\pi}{3\sqrt{3}a}, 0)$. These two points in the BZ are known as two valleys (K and K').



(a) (b)

FIGURE 2 – (a) The honeycomb lattice is a triangular Bravais lattice (with lattice vectors $\vec{a}_1$, $\vec{a}_2$) with two atoms (N and B) in the unit cell (the two-atom basis is indicated by a dashed ellipse). Nearest neighbor vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ are also defined. (b) The corresponding hexagonal Brillouin zone (remarkable points Γ, K and K' are indicated).

4. By expanding the single electron Hamiltonian $H(\vec{k})$ at low energy in the vicinity of $+\vec{K}$ and $-\vec{K}$ (with the notation $\vec{k} = \vec{p} + \xi\vec{K}$, $\xi = \pm$ being the valley index), show that the effective theory becomes (with $\hbar = 1$)

$$\begin{aligned} H(\vec{k}) &\approx \sum_{\xi=\pm} H(\vec{k} = \vec{p} + \xi\vec{K}) \approx \sum_{\xi=\pm} H_\xi(\vec{p}) \\ \text{with } H_\xi &= v_{\mathrm{F}}\, \vec{\alpha}_\xi \cdot \vec{p} + m v_{\mathrm{F}}^2 \beta \end{aligned} \tag{3}$$

where $\vec{\alpha}_\xi = (\xi\sigma_x, -\sigma_y)$ and $\beta = \sigma_z$ in terms of the Pauli matrices σ_x, σ_y and σ_z. Give the expression of v_{F} and $m v_{\mathrm{F}}^2$ as a function of the tight-binding model parameters. Estimate v_{F} and give an order or magnitude for m both in S.I. units.

In the following, we use units such that $v_{\mathrm{F}} = 1$ in addition to $\hbar = 1$.

5. The single electron Hamiltonian (3) may be rewritten in position representation by identifying $\vec{p}$ with $-i\vec{\nabla}$. Show that this gives a Dirac Hamiltonian for each valley $\xi = \pm$.

6. In order to write the Dirac equation following from (3), introduce the corresponding 2+1 dimensional γ^μ matrices (How many such matrices ? What is their algebra ? Their minimal size ? In the representation of the above questions, give their expression in terms of Pauli matrices). What is the physical meaning of the components of the Dirac spinor $\Psi(x)$ here ?

7. By analogy with the situation in 3+1 dimension, we define γ^5 by $\gamma^5 = \frac{i}{3!}\epsilon_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho$. Compute this matrix and its anticommutator with γ^μ. Does it define a chirality transformation which differs from a $U(1)$ phase transformation ? Conclude on the existence of chirality in 2+1 dimension.

8. The system is now placed in a constant and uniform magnetic field $\vec{B}$ perpendicular to the BN layer. How is the Dirac Hamiltonian (3) affected by the magnetic field ? Write also the corresponding Dirac equation.

9. We choose a gauge such that $A^\mu = (A^0, A^1, A^2) = (0, 0, Bx)$ with $B > 0$. Using the fact that the Hamiltonian commutes with p_y, obtain its spectrum and show that the energy of the n^{th} Landau level is given by :

$$\epsilon_n = \pm\sqrt{m^2 + 2neB} \text{ if } n = 1, 2, 3, \ldots \text{ and } \epsilon_0 = -m\xi \text{ if } n = 0 \tag{4}$$

References :

G.W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).
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