Exam for Symmetries and QFT

January 2017.

1. Axion electrodynamics for 3+1 topol. insulators

1) $d^4x' = \det\Lambda \,.\, d^4x = d^4x$ is a Lorentz scalar; $F^{\mu\nu}F_{\mu\nu}$ is a Lorentz scalar $\Big\}$ S_0 is a Lorentz scalar

With the identity, $\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma\right)' = \epsilon^{\mu\nu\rho\sigma}(\det\Lambda)\,\partial_\mu A_\nu \partial_\rho A_\sigma = \pm\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma$ is a Lorentz pseudo-scalar. Therefore S_θ is a Lorentz pseudo-scalar.

2) $S = \int d^4x \left\{ \dfrac{\vec{E}^2 - \vec{B}^2}{2} - \dfrac{\alpha}{\pi}\theta\,\vec{E}\cdot\vec{B} \right\}$ is gauge invariant (it only depends on $\vec{E}$ and $\vec{B}$).

$S_\theta = \int d^4x \, \dfrac{\alpha}{2\pi} \times \dfrac{1}{2} F_{\mu\nu}\tilde{F}^{\mu\nu}$

3) $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = 0 \qquad \dfrac{\partial \mathcal{L}_0}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \qquad \dfrac{\partial \mathcal{L}_\theta}{\partial(\partial_\mu A_\nu)} = \dfrac{\alpha}{2\pi}\,2\theta\,\epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta = \dfrac{\alpha}{\pi}\theta\,\tilde{F}^{\mu\nu}$

EL $\rightarrow \quad \partial_\mu F^{\mu\nu} = \dfrac{\alpha}{\pi}\theta\,\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad$ as $\quad \partial_\mu \tilde{F}^{\mu\nu} = 0$ (consequence of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$)

Maxwell eqts are not modified: $\operatorname{div}\vec{E} = 0$, $\vec{\operatorname{curl}}\,\vec{B} - \partial_t\vec{E} = 0$, $\operatorname{div}\vec{B} = 0$, $\vec{\operatorname{curl}}\,\vec{E} + \partial_t\vec{B} = 0$.

4) $S_\theta = \dfrac{\alpha}{2\pi}\theta\,\epsilon^{\mu\nu\rho\sigma}\int_\Pi d^4x\,\partial_\mu\left(A_\nu\partial_\rho A_\sigma\right) = \dfrac{\alpha}{2\pi}\theta\,\epsilon^{\mu\nu\rho\sigma}\int_{\partial\Pi} d^3S_\mu\, A_\nu\partial_\rho A_\sigma \rightarrow$ do not change the EL eqs.

5) Parity: $\vec{E}\rightarrow -\vec{E}$ and $\vec{B}\rightarrow\vec{B}$. Time reversal: $\vec{E}\rightarrow\vec{E}$ and $\vec{B}\rightarrow -\vec{B}$. Therefore S_0 is invariant under P, T and PT. S_θ is invariant under PT but changes sign under P or T.

6) $S = S_0 + S_\theta \xrightarrow{T} S_0 - S_\theta \qquad$ TRS means $S_0 + S_\theta = S_0 - S_\theta \Rightarrow S_\theta = -S_\theta \Rightarrow \theta = -\theta$

but θ is defined modulo $2\pi \rightarrow \quad \theta = 0$ or $\theta = \pi$ (topol. insulator).

If parity invariance, similarly $\theta = 0$ or π. If neither P nor T invariance, θ is anything in $[0;2\pi[$.

7) EL $\rightarrow \quad \partial_\mu F^{\mu\nu} = \dfrac{\alpha}{\pi}\partial_\mu\left(\theta\tilde{F}^{\mu\nu}\right) = \dfrac{\alpha}{\pi}\tilde{F}^{\mu\nu}\partial_\mu\theta \quad$ as $\quad \partial_\mu\tilde{F}^{\mu\nu} = 0$

Therefore: $\operatorname{div}\vec{E} = \dfrac{\alpha}{\pi}\vec{B}\cdot\vec{\nabla}\theta$, $\vec{\operatorname{curl}}\,\vec{B} - \partial_t\vec{E} = -\dfrac{\alpha}{\pi}\left(\vec{B}\partial_t\theta + \vec{\nabla}\theta\times\vec{E}\right)$, $\operatorname{div}\vec{B} = 0$, $\vec{\operatorname{curl}}\,\vec{E} + \partial_t\vec{B} = 0$

8) $J^\mu = -\dfrac{\delta S_\theta}{\delta A_\mu} = \dfrac{\alpha}{\pi}\,\epsilon^{\nu\rho\sigma\mu}(\partial_\nu A_\rho)\,\partial_\sigma\theta = \dfrac{\alpha}{\pi}(\partial_\sigma\theta)\tilde{F}^{\sigma\mu}$

9) When $\vec{B} = 0$ and $\vec{E} = E_y\,\vec{e_y}$, $\quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_y \\ 0 & 0 & 0 & 0 \\ 0 & E_y & 0 & 0 \end{pmatrix} \quad \rightarrow \quad J_x = \dfrac{\alpha}{\pi}\partial_z\theta \,.\, E_y$

$\vec{j} = \int dz\,\vec{J} = \dfrac{\alpha}{\pi}E_y\,\vec{e_x}\int dz\,\partial_z\theta = -\alpha E_y\,\vec{e_x} \quad\rightarrow\quad \sigma_{xy} = -\alpha = -\dfrac{1}{2}2\alpha = -\dfrac{1}{2}\dfrac{e^2}{h}$

half-quantized Hall effect!

10) In the absence of a magnetic field, it is weird to have a Hall effect. In particular, what decides of the direction of the Hall current? Is it $\vec{j} = \pm\alpha E_y\,\vec{e_x}$? In practise, one needs a small breaking of the TRS on the surface to decide whether $j_x = +\alpha E_y$ or $-\alpha E_y$. In other words, in the absence of TRS breaking on the surface, TRS would imply that the Hall current has to vanish by symmetry.

2. Boron nitride and the 2+1 Dirac equation

1) $f(\vec{k}) = -t\displaystyle\sum_{j=1}^{3} e^{-i\vec{k}\cdot\vec{b_j}}$

$\Delta = \epsilon_N > 0$

2) $\epsilon_{\vec{k}}^{\pm} = \pm\sqrt{\Delta^2 + |f(\vec{k})|^2}$

3) $f(\pm\vec{K}) = 0$ and $\vec{K}-(-\vec{K}) \notin$ reciprocal lattice

4) $H(\vec{p}+\xi\vec{K}) = H(\xi\vec{K}) + \vec{p}\cdot\vec{\nabla}_{\vec{k}} H(\xi\vec{K}) = \begin{pmatrix} \Delta & v_F(\xi p_x + i p_y) \\ v_F(\xi p_x - i p_y) & -\Delta \end{pmatrix}$ with $v_F \equiv \frac{3}{2} ta$ and $m v_F^2 = \Delta$

$v_F \sim \frac{ta}{\hbar} \sim 10^5\,m/s$ $\quad m v_F^2 = \Delta \sim 1\,eV \rightarrow m \sim 10^{-29}\,kg$.

5) $H_\xi = \vec{\alpha}_\xi \cdot \vec{p} + m\beta$ is a Dirac Hamiltonian as $\{\alpha^i, \alpha^j\} = 2\delta^{ij}$ with $i,j = 0,1,2$
$\alpha^0 \equiv \beta$.

6) 3 γ^μ matrices of size 2×2 $\quad \gamma^0 = \sigma_z,\ \gamma^1 = -\xi i \sigma_y,\ \gamma^2 = -i\sigma_x$ for each valley ξ
then $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_{2\times 2}$ (Clifford algebra)
$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$ is the Dirac equation for each valley ξ
$\psi(x) = \begin{pmatrix} \psi_N(x) \\ \psi_B(x) \end{pmatrix}$ probability amplitude of being on each sublattice (N or B).

7) $\gamma^5 = \frac{i}{3!} \epsilon_{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho = -\xi \mathbb{1}_{2\times 2}$
$\{\gamma^5, \gamma^\mu\} = -\xi 2\gamma^\mu \neq 0$
There is no chirality in 2+1 dimension.

8) $H_\xi = \vec{\alpha}_\xi \cdot \vec{p} + m\beta \rightarrow \vec{\alpha}_\xi \cdot (\vec{p} - q\vec{A}) + m\beta$ with $\vec{B} = \vec{\mathrm{curl}}\,\vec{A}$, $q =$ charge of an electron $= -e < 0$
$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \rightarrow (i\gamma^\mu D_\mu - m)\psi(x) = 0$ with $D^\mu \equiv \partial^\mu - iqA^\mu$

9) $\vec{B} = \vec{\nabla}\times\vec{A} = B\vec{e_z}$ with $A^\mu = (0, 0, Bx)$
$H_\xi = \xi \sigma_x p_x - \sigma_y (p_y + eBx) + m\sigma_z$ with $[x, p_x] = i$
$[p_y, H_\xi] = 0 \Rightarrow p_y$ can be taken as a number

$H_\xi = \xi\sigma_x p_x - \sigma_y eB(x + \frac{p_y}{eB}) + m\sigma_z = \xi\sigma_x p_x - \sigma_y eB\tilde{x} + m\sigma_z$ with $[\tilde{x}, p_x] = i$
Let $a \equiv \frac{p_x - ieB\tilde{x}}{\sqrt{2eB}}$ and $a^+ \equiv \frac{p_x + ieB\tilde{x}}{\sqrt{2eB}}$ such that $[a, a^+] = 1$ when $\xi = +1$
$H_+ = \sqrt{2eB} \begin{pmatrix} \frac{m}{\sqrt{2eB}} & a^+ \\ a & -\frac{m}{\sqrt{2eB}} \end{pmatrix} \rightarrow H_+^2 \rightarrow \varepsilon = \pm\sqrt{m^2 + n2eB}$ if $n \in \mathbb{N}^*$
$= -m$ if $n = 0$

For $\xi = -1$ one finds $H_- = \sqrt{2eB} \begin{pmatrix} \frac{m}{\sqrt{2eB}} & a \\ a^+ & -\frac{m}{\sqrt{2eB}} \end{pmatrix} \rightarrow \varepsilon = \pm\sqrt{m^2 + n2eB}$ if $n \in \mathbb{N}^*$
$= +m$ if $n = 0$